

QUANTIZATION DOMAINS

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ABSTRACT. We study the quantization of certain classical field theories using reflection positivity. We give elementary conditions that ensure the resulting vacuum state is cyclic for products of quantum field operators, localized in a bounded Euclidean space-time region \mathcal{O} at positive time. We call such a domain a *quantization domain* for the classical field. The fact that bounded regions are quantization domains in classical field theory is similar to the “Reeh-Schlieder” property in axiomatic quantum field theory.

1. QUANTIZATION DOMAINS

The Reeh-Schlieder property of quantum field theory states that one can recover all the properties described by the field, or by bounded functions of the field called observables, from information localized in any open (bounded) space-time domain \mathcal{O} . The Reeh-Schlieder theorem states that any Wightman field theory, or any Haag-Kastler local quantum theory, has this property [8, 9, 4]. This result is a consequence of analyticity properties that follow in turn from the positivity of the energy and Lorentz covariance, as well as locality.

Here we consider the analog of this result in the context of a classical field on a Euclidean space-time X . We assume that the classical field can be quantized by using the Osterwalder-Schrader property of reflection positivity. We say that a domain \mathcal{O} for the classical field is a *quantization domain*, if the quantization of fields supported in \mathcal{O} gives a dense set of vectors in the physical Hilbert space.

Of course, the are well-known equivalence theorems for Wightman theory and Osterwalder-Schrader (OS) theory ensure that any bounded, positive-time domains in an OS theory is a quantization domain. However, we are interested in investigating field theories for which all the standard axioms may not apply or cases in which they have not been verified. Thus we pose weaker assumptions about the classical theory than the full OS axioms. For example, we replace the relativistic spectrum condition with a weaker assumption.

1.1. Quantization. Consider $d = s + 1$ dimensional space-times

$$X = \mathbb{R} \times \Sigma ,$$

where $\Sigma = X_1 \times \cdots \times X_s$ and $X_j = \mathbb{R}$ or $X_j = S^1$. Call the variable $t \equiv x_0 \in \mathbb{R}$ the time coordinate.

Now consider the Fock space $\mathcal{E}(X)$ over the one-particle space $L_2(X)$. Denote the Fock vacuum vector by $\Omega_0^E \in \mathcal{E}(X)$ and assume that the classical scalar field

$$\Phi(f) = \int \Phi(x)f(x)dx, \quad f \in C_0^\infty(X),$$

defined in terms of the usual creation and annihilation operators, gives rise to the characteristic functional

$$S(f) = \langle \Omega_0^E, e^{i\Phi(f)} \Omega_0^E \rangle = \sum_{n=0}^{\infty} \frac{i^n}{n!} \langle \Omega_0^E, \Phi(f)^n \Omega_0^E \rangle,$$

which is convergent for $f \in C_0^\infty(X)$. Assume that the abelian space-time translation group $T(a)$ and the time reflection ϑ act covariantly on the field and leave the characteristic functional $S(f)$ invariant.

Divide X into a union of three disjoint parts

$$X = X_- \cup X_0 \cup X_+,$$

with the reflection ϑ leaving X_0 invariant and interchanging X_\pm . Let $\mathcal{E}_{\pm,0} \subset \mathcal{E}$ denote the subspace of finite linear combinations

$$A = \sum_{j=1}^N c_j e^{i\Phi(f_j)} \Omega_0^E, \quad f_j \in C_0^\infty(X_\pm),$$

and let \mathcal{E}_\pm denote its closure in \mathcal{E} . Now define equivalence classes

$$(1) \quad \hat{A} = \{A + N\},$$

with N in the null space of the reflection-positive form

$$\langle \hat{A}, \hat{B} \rangle_{\mathcal{H}} = \langle A, \Theta B \rangle_{\mathcal{E}} \quad \text{on } \mathcal{E}_+ \times \mathcal{E}_+.$$

The range of the map (1) defines the pre-Hilbert space \mathcal{H} , whose closure gives the quantization map $A \mapsto \hat{A} \in \mathcal{H}$.

1.2. Standard Hypotheses C1–C3 on the Classical Fields. Denote the Hamiltonian by H , the momentum by \vec{P} , and the time-zero field (averaged with a real test function h depending on the spatial variable) by $\varphi(0, h) = \hat{\Phi}(0, h)$. Let $\|h\|_\alpha$ denote some Schwarz-space norm of the function h depending on the spatial variables, and set

$$(2) \quad \|f\|_{\alpha,1} = \int \|f(t, \cdot)\|_\alpha dt.$$

Assume that

C1 $S(f)$ is space-time translation and time-reflection invariant,

$$S(f_x) = S(\vartheta f) = S(f).$$

C2 $S(f)$ is reflection-positive on X_+ .

C3 There is a constant M , an integer n , and a norm (2) such that

$$|S(f)| \leq e^{M\|f\|_{\alpha,1}^n}.$$

We also assume that time translation in the positive-time direction maps $C_0^\infty(X_\pm)$ into itself. Then the unitary time translation quantizes to a self-adjoint contraction semigroup e^{-tH} generated by the Hamiltonian H . Furthermore, we assume spatial translation in the j^{th} coordinate direction is also unitary and leaves Ω_0^E invariant, so it quantizes to a unitary group $U(x_j) = e^{-ix_j P_j}$ on \mathcal{H} . Let \vec{P} denote the vector with components P_j . Furthermore, the classical field Φ quantizes to an imaginary time field φ .

1.3. Standard Hypotheses on Quantum Mechanics of Fields.

We assume a weak form of the spectral condition: there is a constant $M < \infty$ such that

$$(3) \quad 0 \leq H, \quad \text{and} \quad \pm |\vec{P}| \leq M(H + I).$$

Furthermore, we assume that the field at time zero can be bounded by the energy. In particular, there is a Schwartz-space norm $\|h\|_\alpha$ on the space of time-zero test functions such that the field operators satisfy the form estimates

$$(4) \quad \pm \varphi(0, h) \leq \|h\|_\alpha (H + I).$$

Field Operators. Let $\mathcal{D} = \cup_{0 < \epsilon} e^{-\epsilon H^2} \mathcal{H}$, and note that the vacuum state $\Omega \in \mathcal{D}$. The real-time field $\varphi(x)$ and the imaginary-time field $\varphi_I(x)$ are defined as sesqui-linear forms on $\mathcal{D} \times \mathcal{D}$ by

$$\varphi(x) = e^{itH - i\vec{x} \cdot \vec{P}} \varphi(0) e^{-itH + i\vec{x} \cdot \vec{P}},$$

and

$$\varphi_I(x) = e^{-tH - i\vec{x} \cdot \vec{P}} \varphi(0) e^{tH + i\vec{x} \cdot \vec{P}}.$$

Thus $\varphi_I(x) = \varphi(it, \vec{x})$.

Proposition 1 (Local Field). *Assume the standard hypotheses of §1.2, and let $f \in C_0^\infty$ be real. Then:*

- (1) *The sesqui-linear form $\varphi(f) = \int \varphi(x) f(x) dx$ uniquely defines a symmetric operator $\varphi(f)$ with domain \mathcal{D} , and $\varphi(f)\mathcal{D} \subset \mathcal{D}$.*
- (2) *The operator $\varphi(f)$ extends by continuity to $\mathcal{D}(H)$, and there is a constant $M_1 < \infty$ such that*

$$\|\varphi(f)(H + I)^{-1}\|_{\mathcal{H}} \leq M_1 \|f\|_{\alpha, 1}.$$

- (3) *The closure $\varphi(f)^-$ of the operator $\varphi(f)$ is self-adjoint.*
- (4) *Suppose f and g are real valued functions in C_0^∞ and*

$$[\varphi(f), \varphi(g)]\mathcal{D} = 0.$$

Then the bounded functions of $\varphi(f)^-$ commute with the bounded functions of $\varphi(g)^-$.

1.4. Regularization. Let the x_j 's denote the coordinates of the vector $x \in \mathbb{R}^d$ with $d = s + 1$, and write

$$D^k = \frac{\partial^{k_0+k_1+\dots+k_s}}{\partial x_0^{k_0} \partial x_1^{k_1} \dots \partial x_s^{k_s}}, \text{ and } D_{\vec{x}}^k = \frac{\partial^{k_1+\dots+k_s}}{\partial x_1^{k_1} \dots \partial x_s^{k_s}} \text{ if } k_0 = 0.$$

Let $|k| = k_0 + k_1 + \dots + k_s$ and $k! = k_0!k_1!\dots k_s!$. Likewise, set $(\text{Ad})_A(B) = [A, B]$ and

$$(\text{Ad}_P)^k = (\text{Ad}_H)^{k_0} (\text{Ad}_{P_1})^{k_1} \dots (\text{Ad}_{P_s})^{k_s}.$$

With this notation,

$$(D^k \varphi_I)(x) = (-1)^{k_0} (-i)^{|k|-k_0} (\text{Ad}_P)^k (\varphi_I(x)).$$

Definition 1. The heat-kernel regularized field $\varphi_\epsilon(x)$ and the corresponding heat-kernel regularized imaginary-time field $\varphi_{I,\epsilon}(x)$ are

$$\varphi_\epsilon(x) = e^{-\epsilon H} \varphi(x) e^{-\epsilon H} \text{ and } \varphi_{I,\epsilon}(x) = e^{-\epsilon H} \varphi_I(x) e^{-\epsilon H}, \quad 0 < \epsilon.$$

Theorem 1. Assume the standard hypotheses of §1.2. Then $\varphi_\epsilon(x)$ is a bounded operator on \mathcal{H} for real x , and it extends to a holomorphic, operator-valued function $\varphi_\epsilon(z)$ with $z = (z_0, \vec{z})$ in the domain

$$\mathcal{K}_\epsilon = \{z \in \mathbb{C}^d : |\Im z_0| + |\Im \vec{z}| < \frac{\epsilon}{4M}\}.$$

In this domain, $\varphi_{I,\epsilon}(z) = \varphi_\epsilon(iz_0, \vec{z})$. Also

$$(5) \quad \|D^k \varphi_{I,\epsilon}(z)\| \leq M_1 \left(\frac{4M}{\epsilon}\right)^{|k|+\gamma} (|k| + \gamma)!$$

for all $z \in \mathcal{K}_\epsilon$ and all $k \in \mathbb{N}_0^d$, where $M_1 < \infty$ was introduced in (2) and γ is an integer. Both M_1 and γ depend only on M, α, d .

Proof. Let h be a C_0^∞ test function depending only on the $s = d - 1$ spatial variables. Then the field is an operator-valued distribution for $|t| < \epsilon$ and one can integrate by parts to obtain the identity

$$\begin{aligned} i^{|k|-k_0|} (D^{k_0} \varphi_{I,\epsilon})(t, D_{\vec{x}}^{k-k_0} h) &= (D^k \varphi_{I,\epsilon})(t, h) \\ &= (-1)^{k_0} (-i)^{|k|-k_0} (\text{Ad}_P)^k (\varphi_{I,\epsilon}(t, h)). \end{aligned}$$

One can expand the commutators $(\text{Ad}_P)^k (\varphi_{I,\epsilon}(t, h))$ into $2^{|k|}$ terms $C_{i,\epsilon,t}$. Each such term is exactly $|k|$ multilinear in the components of P , namely the Hamiltonian H and the $s = d - 1$ components P_j , $j = 1, \dots, s$, of the spatial momentum. Suppose that in $C_{i,\epsilon,t}$ exactly α_i of these factors occur to the left of $\varphi_{I,\epsilon}$ and $\beta_i = |k| - \alpha_i$ occur to the right. As H and the components P_j of P mutually commute, each

term $C_{i,\epsilon,t}$ satisfies a bound related to a power of H . Use the assumption (3) and (4), and assume that $1 \leq M$. Then, for $0 \leq t < \epsilon/2$,

$$\begin{aligned}
\|C_{i,\epsilon,t}\| &\leq \sup_i \|e^{-\epsilon H/2} (H+I)^{(\alpha_i+\frac{1}{2})}\| \times \\
&\quad \times \left\| (H+I)^{-(\alpha_i+\frac{1}{2})} C_{i,0,0} (H+I)^{-(\beta_i+\frac{1}{2})} \right\| \times \\
&\quad \times \|e^{-\epsilon H/2} (H+I)^{(\beta_i+\frac{1}{2})}\| \\
&\leq \sup_i \left(\frac{2}{\epsilon}\right)^{\alpha_i+\beta_i+1} (\alpha_i + \beta_i + 1)! M^{|k|} \times \\
&\quad \times \|(H+I)^{-1/2} \varphi(0, h) (H+I)^{-1/2}\| \\
&\leq \left(\frac{2M}{\epsilon}\right)^{|k|+1} (|k| + 1)! \|h\|_\alpha .
\end{aligned}$$

As \mathcal{O} is a fixed bounded open set, any Schwartz-space function is in $C_0^\infty(\mathcal{O})$ and any Schwartz-space norm satisfies

$$\|h\|_\alpha \leq M_2 \|(-\Delta_{\vec{x}} + 1)^r h\|_{L_2(\mathbb{R}^{d-1})} ,$$

for some constants r and M_2 that depend on the original norm and on \mathcal{O} . Write

$$h = (-\Delta_{\vec{x}} + 1)^{r+[(d+1)/2]} (-\Delta_{\vec{x}} + 1)^{-r-[(d+1)/2]} h ,$$

with $[\cdot]$ denoting the integer part. One can take the derivatives in each term in the multinomial expansion of $(-\Delta_{\vec{x}} + 1)^{r+[(d+1)/2]}$ to be the derivatives $D^{k'}$. Then, using

$$\|(-\Delta_{\vec{x}} + 1)^{-r-[(d+1)/2]} h\|_\alpha \leq M_3 \|(-\Delta_{\vec{x}} + 1)^{-(d+1)/2} h\|_{L_2(\mathbb{R}^{d-1})} ,$$

one has, with a new constant M_3 , and with $\gamma = 2r + d + 2$,

$$\begin{aligned}
\|e^{-\epsilon H} (D^k \varphi_I)(0, h) e^{-\epsilon H}\| &\leq 2^{|k|} M_3 \left(\frac{2M}{\epsilon}\right)^{|k|+\gamma} (|k| + \gamma)! \times \\
&\quad \times \|(-\Delta_{\vec{x}} + 1)^{-(d+1)/2} h\|_{L_2} .
\end{aligned}$$

For spatial dimension s , the Dirac measure $\delta_{\vec{x}}$ is an element of the Sobolev space with norm

$$\|h\|_{H^{-(d+1)/2}} = \|(-\Delta_{\vec{x}} + 1)^{-(s+2)/2} h\|_{L_2(\mathbb{R}^s)} .$$

Thus we can take a limit $k \rightarrow \infty$ of C^∞ -functions h_k supported in \mathcal{O} and converging to $\delta_{\vec{x}}$ for $\vec{x} \in \mathcal{O}$. As a consequence, the operator $e^{-\epsilon H} (D^k \varphi_I)(0, h) e^{-\epsilon H}$ extends by continuity to $h = \delta_{\vec{x}}$. Likewise the same bound holds for the extension, so with a new constant M_4 ,

$$\|e^{-\epsilon H} (D^k \varphi_I)(0, \vec{x}) e^{-\epsilon H}\|_{\mathcal{H}} \leq M_4 \left(\frac{4M}{\epsilon}\right)^{|k|+\gamma} (|k| + \gamma)! .$$

This is the claimed bound (5) on the heat-kernel regularized field with $x \in \mathcal{O}$, with the constant $M_1 = M_4$. As the momentum \vec{P} and H commute, and as time-translation is unitary on \mathcal{H} , this bound also holds for arbitrary spatial points \vec{x} .

As a consequence of this estimate in the derivatives of the field, the heat-kernel regularized, imaginary-time field is real-analytic in a neighbourhood of $|t| < \epsilon/(4M)$ and for all $\vec{x} \in \mathbb{R}^d$. In fact, the power series for an analytic continuation converges in a neighbourhood of zero, with radius of convergence $\epsilon/(4M)$. The unitary action of spatial translations, that commute with H , leads to analyticity for all real \vec{x} . \square

Lemma 1. *Assume the hypotheses of Theorem 1. Let $\widehat{A}, \widehat{B} \in \mathcal{H}$, and $0 < \epsilon$. Let M, M_1 , and γ be the constants in the theorem cited, and define*

$$F(x) = \langle \widehat{A}, e^{-\epsilon H} \varphi_I(x) e^{-\epsilon H} \widehat{B} \rangle_{\mathcal{H}}.$$

Then $F(x)$ is real-analytic for $|t| < \epsilon/(4M)$ and all real \vec{x} , and $F(x)$ extends to a holomorphic function in the domain $z = (z_0, \vec{z})$ with $|z_0| + |\Im \vec{z}| < \epsilon/(4M)$. Furthermore, the analytic continuation to this domain satisfies the uniform bound

$$|F(z)| \leq \frac{M_1}{\epsilon^\gamma} \|\widehat{A}\|_{\mathcal{H}} \|\widehat{B}\|_{\mathcal{H}}.$$

Proof. One can bound $|D^k F(x)| \leq \|e^{-\epsilon H} D^k \varphi_I(x) e^{-\epsilon H}\| \|\widehat{A}\|_{\mathcal{H}} \|\widehat{B}\|_{\mathcal{H}}$ using the operator norm bounds of Theorem 1. This shows that the power series for $F(x)$ converges absolutely in the desired domain. The bound (5) then yields the uniform estimate. \square

1.5. Quantization Domain. For $\mathcal{O} \subset X_+$, let $\mathcal{A}(\mathcal{O})$ denote the algebra of polynomials in fields averaged with C^∞ -functions supported in \mathcal{O} .

Definition 2. We call $\mathcal{O} \subset \mathbb{R}_+^d$ a *quantization domain*, if the quantization map $A \mapsto \widehat{A}$ given in Equ. (1) takes the linear subspace $\mathcal{D}(\mathcal{O}) = \mathcal{A}(\mathcal{O})\Omega_0^{\mathbb{E}}$ into a subspace $\widehat{\mathcal{D}(\mathcal{O})} \subset \mathcal{H}$ that is dense in \mathcal{H} .

Theorem 2 (Non-trivial Quantization Domains). *With the hypotheses of §1.2, any open set $\mathcal{O} \subset X_+$ is a quantization domain.*

Proof. If \mathcal{O}' is a quantization domain, then so is any larger set $\mathcal{O} \supset \mathcal{O}'$; therefore it is no loss of generality to take \mathcal{O} to be a bounded, open set of small diameter. Let $\chi \in \mathcal{H}$ be any vector that is orthogonal to \widehat{B} for all $B \in \mathcal{D}(\mathcal{O})$, i.e.,

$$(6) \quad \langle \chi, \widehat{B} \rangle_{\mathcal{H}} = 0 \quad \text{for all } B \in \mathcal{D}(\mathcal{O}).$$

The theorem holds if and only if such a vector χ must vanish, so we now show that $\chi = 0$.

It is sufficient to consider $A \in \mathcal{A}(\mathcal{O})$ of the form $A_n(f_1, \dots, f_n) = \Phi(f_1) \cdots \Phi(f_n)$, with each f_j supported in the domain \mathcal{O} , and with arbitrary n . Define $A_n(x_1, \dots, x_n) = \Phi(x_1) \cdots \Phi(x_n)$, where now the subscript labels different d -vectors. Set $B_n = A_n \Omega_0^{\mathbb{E}}$ and denote the quantization of B_n by $\widehat{B}_n(x_1, \dots, x_n)$. Note both A_n and \widehat{B}_n are symmetric

under permutations of the coordinates $x_1, \dots, x_n \rightarrow x_{\pi_1}, \dots, x_{\pi_n}$, for $\pi \in S_n$. Also

$$\widehat{A}_n(f_1, \dots, f_n) = \int \widehat{A}_n(x_1, \dots, x_n) f_1(x_1) \cdots f_n(x_n) dx_1 \cdots dx_n$$

satisfies

$$(7) \quad \langle \chi, \widehat{B}_n(f_1, \dots, f_n) \rangle_{\mathcal{H}} = 0, \quad B_n(f_1, \dots, f_n) = A_n(f_1, \dots, f_n) \Omega_0^E.$$

The vector $\widehat{B}_n(x_1, \dots, x_n)$ is the anti-time-ordered product of the imaginary-time fields,

$$(8) \quad \widehat{B}_n(x_1, \dots, x_n) = \varphi_I(x_{i_1}) \cdots \varphi_I(x_{i_n}) \Omega, \quad \text{where } t_{i_1} \leq \cdots \leq t_{i_n}.$$

As (8) is symmetric under permutations of the coordinates, we need only consider the case $t_1 < \cdots < t_n$, which we now assume. Furthermore, we can choose $0 < \epsilon$ sufficiently small, so that each f_j has support in the set $\mathcal{O}_j \subset \mathcal{O}$ with \mathcal{O}_j lying in the time-strip $[T + 3j\epsilon, T + 3(j+1)\epsilon]$, for $j = 1, \dots, n$. This ensures a time-separation $t_{j+1} - t_j \geq 2\epsilon$ between x_{j+1} and x_j for each j . Hence for $x_j \in \mathcal{O}_j$, Theorem 1 ensures that $\widehat{B}_n(x_1, \dots, x_n)$ is a real analytic function of x_1, \dots, x_n and it extends to a holomorphic function in the complex domain $|z_{j,0}| + |\Im \bar{z}_j| < \epsilon/(4M)$ for $j = 1, \dots, n$. In this domain the extension satisfies the uniform bound

$$\|\widehat{B}_n(z_1, \dots, z_n)\|_{\mathcal{H}} \leq \left(\frac{M_1}{\epsilon^\gamma}\right)^n.$$

For each j fixed take a sequence of C_0^∞ functions $f_{j,\ell}$ for $\ell = 1, \dots$, converging to the delta function, i.e., $f_{j,\ell} \rightarrow \delta_{x_j}$ with $x_j \in \mathcal{O}_j$. Then the property (7) ensures that the holomorphic function

$$F(z_1, \dots, z_n) = \langle \chi, \widehat{B}_n(z_1, \dots, z_n) \rangle_{\mathcal{H}}$$

vanishes in the analyticity domain $|z_{j,0}| + |\Im \bar{z}_j| < \epsilon(4M)^{-1}$, and hence in any analytic continuation of this domain.

The symmetry of $\widehat{B}_n(x_1, \dots, x_n)$ under $x_j \rightarrow x_{\pi_j}$ for $\pi \in S_n$ shows that the analyticity and the bound on $F_n(z_1, \dots, z_n)$ extends to the domain obtained by the permutation $z_j \rightarrow z_{\pi_j}$. Similarly, analyticity and the uniform bound both extend to the union of these domains over $T > 0$. These domains include all real points (x_1, \dots, x_n) with positive, non-coinciding times. \square

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